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From stochastic quantization to bulk quantization: Schwinger-Dyson equations and S-matrix

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Abstract: In stochastic quantization, ordinary 4-dimensional Euclidean quantum field theory is expressed as a functional integral over fields in 5 dimensions with a fictitious 5th time. However a broader framework, which we call "bulk quantization", is required for extension to fermions, and for the increased power afforded by the higher symmetry of the 5-dimensional action that is topological when expressed in terms of auxiliary fields. Within the broader framework, we give a direct proof by means of Schwinger-Dyson equations that a time-slice of the 5-dimensional theory is equivalent to the usual 4-dimensional theory. The proof does not rely on the conjecture that the relevant stochastic process relaxes to an equilibrium distribution. Rather, it depends on the higher symmetry of the 5-dimensional action which includes a BRST-type topological invariance, and invariance under translation and inversion in the 5-th time. We express the physical S-matrix directly in terms of the truncated 5-dimensional correlation functions, for which "going off the mass-shell" means going from the 3 physical degrees of freedom to 5 independent variables. We derive the Landau-Cutokosky rules of the 5-dimensional theory which include the physical unitarity relation.

Keywords: Renormalization Regularization and Renormalons, BRST Quantization, Non-perturbative Effects, QCD.

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Contents

1. Introduction

There are a number of cases where it is helpful to increase the number of dimensions above what seems to be required. An example is the use of complex numbers, with an "imaginary" component, to solve a problem involving real numbers. There is also Feynman's expression of 3-dimensional quantum mechanical perturbation theory by 4-dimensional space-time integrals. For the quantization of gauge fields, it is useful to repeat Feynman's idea and express ordinary "4-dimensional" quantum field theory in terms of a functional integral in five dimensions, which allows one to overcome the Gribov problem [1, 2]. Use of the 5th dimension in quantum field theory began with the paper of Parisi and Wu [1] which introduced a method that has become known as stochastic quantization. Here one regards the 4-dimensional euclidean probability distribution $N\exp[-S(\phi)]$ as the equilibrium Boltzmann distribution to which a stochastic process relaxes. Such a process is described by the Fokker-Planck equation:

$$\dot{P} = \frac{\partial}{\partial\phi_i}\left[\left(\frac{\partial}{\partial\phi_i} + \frac{\partial S}{\partial\phi_i}\right)P\right], \tag{1.1}$$

where $P = P(\phi, t)$ is a probability that evolves in a "fictional" or "fifth" time t. In the notation used here, the discrete index i represents the usual 4-dimensional euclidean space-time variable x_μ, with $\mu = 1, \ldots, 4$, and possibly other labels, and for the field variable we write ϕ_i instead of $\phi(x)$. More generally we have

$$\dot{P} = \frac{\partial}{\partial \phi_i} \left[K_{ij}(\phi) \left(\frac{\partial}{\partial \phi_j} + \frac{\partial S}{\partial \phi_j} \right) P \right], \tag{1.2}$$

where $K_{ij}(\phi)$ is an appropriate kernel. One sees immediately that the desired euclidean distribution $N \exp[-S(\phi)]$ is a time-independent solution of this equation.

Stochastic quantization relies on the assumption that starting from any initial normalized probability distribution $P(\phi, 0) = P_0(\phi)$, the solution $P(\phi, t)$ relaxes to an equilibrium distribution $P_{\mathrm{eq}}(\phi)$ which moreover is unique,

$$\lim_{t \to \infty} P(\phi, t) = P_{\mathrm{eq}}(\phi) = N \exp[-S(\phi)], \tag{1.3}$$

and stochastic quantization is realized concretely in numerical simulations of quantum field theory by the Monte-Carlo method, where the 5th time corresponds to the number of sweeps of the lattice. According to the Frobenius theorem, relaxation to a unique equilibrium distribution does hold for a large class of discrete Markov processes with a finite number of variables. However it is not known whether the Frobenius theorem applies in the field-theoretic context with its infinite number of degrees of freedom.

This approach has been considerably elaborated over the years [3]–[7], and new ideas have emerged. Just as in conventional quantum field theory one does not deal directly with the Schroedinger equation but rather with correlation functions in 4 dimensions calculated from a 4-dimensional functional integral, likewise in stochastic quantization, it is more efficient to deal, not with the Fokker-Planck equation itself, but rather with correlation functions in 5 dimensions calculated from a 5-dimensional functional integral with a 5-dimensional local action $I = \int d^4x dt \mathcal{L}$. It is convenient to let the 5th time run from $t = -\infty$ to $t = +\infty$, and to exploit invariance under time translation and, significantly, under inversion in the 5th time, $t \to -t$, and there are no initial conditions to be specified. (This corresponds to a stationary process that is at equilibrium for all 5th time.) It turns out that for such purposes as renormalization, the most effective functional integral representation is as a topological quantum field theory (i.e. with a BRST-exact action) that necessarily involves auxiliary and ghost fields. These fields do not have an immediate stochastic interpretation. The stochastic interpretation is also lost in the extension of the theory to fermi-dirac fields. Thus the relation to a stochastic and relaxation process is rather tenuous, and one feels the need of a more fundamental approach for quantization with an additional time.

Inspired by the recent developments in holography principles, we have recently revisited stochastic quantization of gauge theory in the context of topological quantum field theory [8, 9]. The beauty of the construction suggests to us to further

JHEP08(2001)016

investigate the guiding principles of quantization with an additional time. In the present article we shall give a direct proof by means of Schwinger-Dyson equations that a time-slice of the 5-dimensional theory is equivalent to the usual 4-dimensional euclidean theory. It holds for fermi as well as bose fields. Our proof does *not* involve any relaxation, but relies instead on the higher symmetry of the 5-dimensional action, including its BRST-type topological invariance, and invariance under translation in the 5th time. Somewhat surprisingly, invariance under inversion in the 5th time, $t \to -t$, plays an essential role in our proof. We also show that symmetry under inversion in the 5th time selects the class of 5-dimensional theories that are equivalent on a time-slice to a local 4-dimensional theory. We shall also express the physical S-matrix directly in terms of the truncated correlation functions of the 5-dimensional theory, eq. (4.31) below. When momentum vectors of physical particles are "taken off the mass shell", they acquire not 4 but 5 components. We also elaborate on the Cutkowski-Landau rules along the same lines, as well as on the conservation laws when there are global symmetries.

Because the connection to a stochastic process is not obvious in the present approach, this name no longer seems appropriate. We call our approach "bulk" quantization. Here we regard the 5-dimensional space with local topological action as the bulk, with the physical 4-space living on a time-slice. In the present article, we deal only with a theory of non-gauge type. In this case there is an exact equivalence of the 4- and 5-dimensional formulations, and we already find quite remarkable that supersymmetry — in reality the consequences of the BRST symmetry — allows one to directly prove the results without having to invoke the Frobenius theorem. This opens interesting questions about the convergence of correlation functions in the limit of equal (fifth) time. For gauge theories, much more must be done, which we will present in a separate publication. Whereas for theories of non-gauge type the equivalence of the standard and bulk quantization is exact, for gauge theories bulk quantization allows a different type a gauge fixing that overcomes the Gribov problem.

JHEP08(2001)016

2. Equivalence of standard and bulk quantization

We consider a theory of a field or set of fields $\phi(x)$ in d euclidean space-time dimensions, for $x = x_\mu$, with $\mu = 1, \ldots, d$, defined by a local action

$$S[\phi] = \int d^d x \; \mathcal{L}_d \, . \tag{2.1}$$

For example, we may take

$$\mathcal{L}_d = \left(\frac{1}{2}\right) (\partial_\mu \phi)^2 + \left(\frac{1}{2}\right) m^2 \phi^2 + \left(\frac{1}{4}\right) g\phi^4 \, . \tag{2.2}$$

Expectation-values are calculated from

$$\langle \mathcal{O} \rangle_d \equiv N \int d\phi \; \mathcal{O} \exp(-S) \,, \tag{2.3}$$

for any observable $\mathcal{O} = \mathcal{O}(\phi)$.

This theory is completely described, at least perturbatively, by the familiar set of Schwinger-Dyson (SD) equations. They follow from the identity

$$\begin{aligned} 0 &= \int d\phi \; \frac{\partial[\mathcal{O}\exp(-S)]}{\partial \phi_i} \,, \\ \left\langle \frac{\partial \mathcal{O}}{\partial \phi_i} - \mathcal{O}\frac{\partial S}{\partial \phi_i} \right\rangle_d &= 0 \end{aligned} \tag{2.4}$$

that holds for all observables $\mathcal{O}(\phi)$. Here and below the discrete index i or j etc. represents the continuous index x_μ, $\mu = 1, \ldots, d$ and any internal indices. It is sufficient to take $\mathcal{O}(\phi) = \exp(j_i \phi_i)$, where the j_i are arbitrary sources, to generate a complete set of Schwinger-Dyson equations that determine the partition function $Z(j) = \langle \exp(j_i \phi_i) \rangle$.

Bulk quantization of the same theory is expressed in terms of a $(d{+}1)$-dimensional theory that involves a quartet of fields $\phi(x,t), \psi(x,t), \bar\psi(x,t), b(x,t)$, with ghost number $0, 1, -1, 0$ respectively, the ghost and anti-ghost fields being fermionic. A topological BRST operator s acts on these fields according to

$$\begin{aligned} s\phi &= \psi \,, & s\psi &= 0 \,, \\ s\bar\psi &= b \,, & sb &= 0 \,, \end{aligned} \tag{2.5}$$

and obviously satisfies $s^2 = 0$. A $(d+1)$-dimensional action is defined by

$$\begin{aligned} I_{\rm tot} &= \int dt \; \frac{1}{2}\dot S + I \,, \\ I &\equiv \int dt \; s\left\{ \bar\psi_i \left[\dot\phi_i + K_{ij}\left(\frac{\partial S}{\partial \phi_j} + b_j \right) \right] \right\} , \end{aligned} \tag{2.6}$$

where $\dot S \equiv \partial S/\partial t$, $\dot\phi \equiv \partial\phi/\partial t$ etc. It is topological in the sense that it is an exact derivative plus an s-exact term. In this section we consistently replace the continuous d-dimensional variable x by the discrete index i, but we keep the continuous variable t, so the fields are represented by $\phi_i(t), \psi_i(t), \bar\psi_i(t), b_i(t)$. The action I has the expansion

$$I = \int dt \left\{ b_i \left[\dot\phi_i + K_{ij}\left(\frac{\partial S}{\partial \phi_j} + b_j \right) \right] - \bar\psi_i \left(\dot\psi_i + L_{il}\psi_l \right) \right\} , \tag{2.7}$$

where

$$L_{il} \equiv K_{ij} \frac{\partial^2 S}{\partial \phi_j \partial \phi_l} + \frac{\partial K_{ij}}{\partial \phi_l} \left(\frac{\partial S}{\partial \phi_j} + b_j \right) . \tag{2.8}$$

In $d = 4$ dimensions, the kernel K_{ij} is severely restricted by renormalizability. For a scalar theory we will have $K = \text{const}$. The motive for introducing the kernel, apart from generality, is that in a theory with a Dirac spinor q it is helpful for convergence of Feynman integrals to take $K = -\gamma_\mu \partial_\mu +$ (non derivative), so that the highest derivative of q, contained in

$$K \frac{\delta S}{\delta \bar{q}} = (-\gamma_\mu \partial_\mu + \cdots)(\gamma_\mu \partial_\mu + \cdots) q = (-\partial^2 + \cdots) q \tag{2.9}$$

is a positive elliptic operator.

Contact with the physical theory in d dimensions is made by requiring that physical observables $\mathcal{O}$ be functions of ϕ that are restricted to a time-slice $t = t_0$. By time-translation invariance we may take $t_0 = 0$. Thus the allowed physical observables are of the form $\mathcal{O} = \mathcal{O}(\phi(0))$. Derivatives with respect to t are not allowed. Note that the topological invariance under reparametrization of the the variable t is also an invariance of the $(d+1)$-dimensional action $\frac{1}{2} \int dt \dot{S}$. We wish to emphasize that observables are *not* of topological type: they are *not* s-exact.

We now come to the essential point. *Statement*: the s-invariant theory in $(d+1)$ dimensions, with action $I_{\rm tot}$ and observables restricted to a time-slice, is identical to the d-dimensional euclidean theory with action S, provided only that the kernel K_{ij} is symmetric and transverse,

$$K_{ij} = K_{ji} \, , \qquad \frac{\partial K_{ij}}{\partial \phi_j} = 0 \, . \tag{2.10}$$

In other words we assert that the two theories give the same expectation values for all physical observables,

$$\langle \mathcal{O}(\phi) \rangle_d = \langle \mathcal{O}(\phi(0)) \rangle_{d+1} \, , \tag{2.11}$$

where $\langle \mathcal{O}(\phi) \rangle_d$ is defined in (2.3) and

$$\langle \mathcal{O} \rangle_{d+1} \equiv N \int \mathcal{D}\phi \mathcal{D}b \mathcal{D}\psi \mathcal{D}\bar{\psi} \; \mathcal{O} \exp I_{\rm tot} \, . \tag{2.12}$$

[Note that the weights are $\exp(-S)$ and $\exp(+I_{\rm tot})$.]

Proof. It is sufficient to establish that the SD eqs. (2.4) are satisfied for the expectation-values $\langle \mathcal{O}(\phi(0)) \rangle_{d+1}$. Since $\mathcal{O}(\phi)$ and $K_{ij}(\phi)$ are independent of b, the integral,

$$0 = \int \mathcal{D}\phi \mathcal{D}b \mathcal{D}\psi \mathcal{D}\bar{\psi} \; \mathcal{O}(\phi(0)) \; K^{-1}_{ki}(\phi(0)) \, \frac{\delta \exp I_{\rm tot}}{\delta b_i(0)} \, , \tag{2.13}$$

vanishes because the integrand is a derivative with respect to b. By (2.7), this gives

$$\begin{aligned} 0 &= \left\langle \; \mathcal{O} \; K^{-1}_{ki} \; \left(\dot{\phi}_i + K_{ij} \frac{\partial S}{\partial \phi_j} + 2 K_{ij} b_j - \bar{\psi}_j \frac{\partial K_{ji}}{\partial \phi_l} \psi_l \right) \right\rangle_{d+1} \\ &= \left\langle \; \mathcal{O} \; \left(K^{-1}_{ki} \; \dot{\phi}_i + \frac{\partial S}{\partial \phi_k} + 2 b_k - K^{-1}_{ki} \; \bar{\psi}_j \frac{\partial K_{ji}}{\partial \phi_l} \psi_l \right) \right\rangle_{d+1} \, , \end{aligned} \tag{2.14}$$

where we have written $\mathcal{O} = \mathcal{O}(\phi(0))$ etc, and it is understood that all fields are evaluated at $t = 0$. We will show in the next section that the reduced action I'_{tot}, obtained after integrating out the ghosts, is invariant under a time reversal transformation under which ϕ transforms according to $\phi_i(t) \to \phi_i(-t)$, so

$$\left\langle \mathcal{O}(\phi(0))\ K^{-1}_{ki}(\phi(0))\ \dot{\phi}_i(0) \right\rangle_{d+1} = 0\,. \tag{2.15}$$

We write $b_j = s\bar{\psi}_j$, and use s-invariance to obtain

$$\begin{aligned} 0 &= \left\langle \mathcal{O} \left(\frac{\partial S}{\partial \phi_k} + K^{-1}_{ki} \frac{\partial K_{ij}}{\partial \phi_l} \psi_l \bar{\psi}_j \right) - 2s\mathcal{O}\ \bar{\psi}_k \right\rangle_{d+1} \\ &= \left\langle \mathcal{O} \frac{\partial S}{\partial \phi_k} - 2\ \frac{\partial \mathcal{O}}{\partial \phi_l} \psi_l \bar{\psi}_k + \mathcal{O} K^{-1}_{ki} \frac{\partial K_{ij}}{\partial \phi_l}\ \ \psi_l \bar{\psi}_j \right\rangle_{d+1} . \end{aligned} \tag{2.16}$$

All fields are evaluated at $t = 0$.

In appendix A we show that inside the expectation value we may make the substitution

$$\psi_j(0)\bar{\psi}_l(0) \longrightarrow \frac{1}{2}\delta_{jl}\,. \tag{2.17}$$

This gives

$$0 = \left\langle \mathcal{O} \frac{\partial S}{\partial \phi_k} - \frac{\partial \mathcal{O}}{\partial \phi_k} + \frac{1}{2} K^{-1}_{ki} \mathcal{O} \frac{\partial K_{ij}}{\partial \phi_j} \right\rangle_{d+1} , \tag{2.18}$$

and by (2.10) we obtain the SD equations of the d-dimensional theory

$$0 = \left\langle \mathcal{O} \frac{\partial S}{\partial \phi_k} - \frac{\partial \mathcal{O}}{\partial \phi_k} \right\rangle_{d+1} , \tag{2.19}$$

which hold for every $\mathcal{O} = \mathcal{O}(\phi(0))$. We have thus proven that the two formulations give the same correlation functions, at the very basic level of Dyson-Schwinger equations.

3. Time reversal, canonical structure, and stability of the action

In dealing with renormalizable theories, one must ask what is the most general action that is compatible with given symmetries and field dimensions? In particular do the symmetries assure that, under renormalization, a topological $(d+1)$-dimensional action remains equivalent to a local d-dimensional euclidean theory? With reference to eqs. (2.6) and (2.7), we see that symmetry must preserve the form of term $b_i K_{ij} (\frac{\partial S}{\partial \phi_j} + b_j)$. Counter-terms cannot be tolerated that would change it, for example, to the form $b_i K_{ij}(M_j + b_j)$, where $M_j(\phi)$ is not derivable from an action. Fortunately such counter-terms are forbidden by the symmetry under time reversal that we already used in the previous section. This is a symmetry of the ϕ–b sector that is obtained by setting to 0 all sources with non-zero ghost number. The symmetry of this sector is controlled by the symmetry of the reduced action that is obtained by integrating out the ghost fields.

Statement: The reduced action $I'_{\rm tot}(b,\phi)$, obtained by integrating out the ghost fields, is local, and invariant under the time reversal transformation

$$\begin{aligned} \phi_i(t) &\longrightarrow \phi_i^{\rm T}(t) = \phi_i(-t)\,, \\ b_i(t) &\longrightarrow b_i^{\rm T}(t) = -b_i(-t) - \frac{\partial S}{\partial \phi_i}(-t)\,. \end{aligned} \tag{3.1}$$

In terms of the variable $b'_i \equiv b_i + \frac{1}{2}\frac{\partial S}{\partial \phi_i}$ which transforms according to

$$b'_i(t) \longrightarrow b'^{T}_i(t) = -b'_i(-t)\,, \tag{3.2}$$

the reduced action is given by

$$I'_{\rm tot}(\phi, b') = \int dt \left\{ b'_i \dot{\phi}_i + b'_i K_{ij} b'_j - \frac{1}{4}\frac{\partial S}{\partial \phi_i} K_{ij} \frac{\partial S}{\partial \phi_j} + \frac{1}{2} K_{ij} \frac{\partial^2 S}{\partial \phi_j \partial \phi_i} \right\}. \tag{3.3}$$

This action has a standard canonical structure, with momentum canonical to ϕ_j given by $p_j = ib'_j$. It is obviously invariant under the above time-reversal transformation. The reduced action and the time-reversal transformation and the canonical change of variable from b to b' are all local.

Proof. By (2.7), the integral over the ghosts gives

$$\int d\psi d\bar{\psi} \exp I_{\rm gh} = \det\left[\frac{\partial}{\partial t} + L(\phi, b)\right], \tag{3.4}$$

where L is given in (2.8). Apart from an irrelevant multiplicative constant, this may be written

$$\begin{aligned} \int d\psi d\bar{\psi} \exp I_{\rm gh} &= \det[1 + G_0 L_{\rm int}(\phi, b)] \\ &= \exp {\rm Tr} \ln[1 + G_0 L_{\rm int}(\phi, b)] \\ &= \exp {\rm Tr} \left[L_{\rm int} G_0 - \frac{1}{2} L_{\rm int} G_0 L_{\rm int} G_0 + \cdots \right]. \end{aligned} \tag{3.5}$$

Here G_0 is the integral operator $G_0 = (\partial/\partial t + L_0)^{-1}$, with kernel $G_{0,ij}(t-u)$, given in (A.7). This kernal is retarded, $G_{0,ij}(t-u) = 0$ for $t < u$, so all terms in the expansion vanish except the first, and we have

$$\begin{aligned} \int d\psi d\bar{\psi} \exp I_{\rm gh} &= \exp\left[\int dt\; L_{{\rm int},ij} G_{0,ji}(t-u)|_{u=t}\right] \\ &= \exp\left[\int dt\; L_{{\rm int},ii}\theta(0)\right] \\ &= \exp\left[\int dt\; \frac{1}{2} L_{{\rm int},ii}\right], \end{aligned} \tag{3.6}$$

where $\theta(t)$ is the step function, and we have used the consistent determination $\theta(0) = 1/2$. By (2.8) and (2.10), this gives

$$\int d\psi d\bar{\psi} \exp I_{\rm gh} = \exp\left(\int dt\ \frac{1}{2} K_{ij} \frac{\partial^2 S}{\partial\phi_j \partial\phi_i}\right). \tag{3.7}$$

Only the tadpole term survives, and the ghost integral contributes a local term to the reduced action that depends only on ϕ. From (2.7), we conclude that the reduced action, obtained by integrating out the ghosts, is given by

$$I'_{\rm tot}(\phi, b) = \int dt \left\{\frac{1}{2}\dot{S} + b_i\left[\dot{\phi}_i + K_{ij}\left(\frac{\partial S}{\partial \phi_j} + b_j\right)\right] + \frac{1}{2} K_{ij} \frac{\partial^2 S}{\partial\phi_j \partial\phi_i}\right\}. \tag{3.8}$$

With $\dot{S} = \frac{\partial S}{\partial \phi_i}\dot{\phi}_i$, eq. (3.3) follows.

For completeness we exhibit the integration over the canonically conjugate field b', which is immediate from (3.3). The integral is gaussian and converges because b' is a purely imaginary field,

$$\begin{aligned}\exp I''_{\rm tot}(\phi) &\equiv \int db'\ \exp I'_{\rm tot}(\phi, b') \\ &= (\det K)^{-1/2} \exp \int dt \left\{-\frac{1}{4}\dot{\phi}_i (K^{-1})_{ij} \dot{\phi}_j - \frac{1}{4}\frac{\partial S}{\partial \phi_i} K_{ij} \frac{\partial S}{\partial \phi_j} + \right. \\ &\qquad \left. + \frac{1}{2} K_{ij} \frac{\partial^2 S}{\partial\phi_j \partial\phi_i}\right\}. \end{aligned} \tag{3.9}$$

It should be noticed that when the kernel K_{ij} is the identity, the time reversal symmetry extends to the ghost part of the action, with the transformation $\psi_i(x,t) \to \bar{\psi}_i(x,-t)$ and $\bar{\psi}_i(x,t) \to \psi_i(x,-t)$. For a generic kernel the ghost part of the action is not invariant under time reversal. However, as we shall show shortly, lack of time-reversal invariance in the ghost sector does not prevent us from using time-reversal invariance in the ϕ–b sector to prove stability of the action under renormalization and convergence of the correlation functions in a slice at fixed time toward those of the ordinary formulation. The lack of the time reversal invariance in the ghost sector seems to be an artifact of the choice of a kernel, and cannot affect physical quantities.

Symmetry under the time reversal selects the $(d+1)$-dimensional theories that are derivable from a d-dimensional action S. Indeed, if the counter-terms generated a generic drift force that is not derivable from an action, $M_j(\phi) \neq \partial S/\partial\phi_j$, then upon integrating out the ghost fields and completing the square as in (3.3), one gets the cross term $(-1/2)\int dt \dot{\phi}_i M_i(\phi)$. This term is not an exact time derivative, and would violate time-reversal invariance unless $M_j(\phi) = \partial S/\partial \phi_j$, for some euclidean action S, in which case it is a mere boundary term $(-1/2)\int dt\dot{S}$. More precisely, if we consider the class of actions

$$s(\bar{\Psi}_i(\partial_t \phi_i + P_i(\phi) + K_{ij} b_j)), \tag{3.10}$$

one gets the t-parity violating term $(-1/2)\int dt\dot{\phi}_i K^{-1}_{ij} P_j(\phi)$ which disappears if and only if the integrand is a boundary term, that is when $P_i(\phi) = K_{ij}\partial S/\partial\phi_j$, which is the desired property. Part of our intuition is actually that the integral $\int dt\dot{\phi}_i \frac{\partial S}{\partial \phi_j}$ gives back the action S in d dimensions, so we might interpret the quantization with an additional time as a very refined version of Stokes theorem, generalized to the case of path integration.

We now sketch the argument that establishes the stability of the action in the context of renormalizable theories. The full renormalized action must be s-exact, and the first step is to construct the most general local renormalized action that is s-exact and allowed by power counting. Next we use the important property of invariance under inversion of the 5th time which is a property of the correlation functions in the b–ϕ sector, namely correlation functions with external ϕ and b legs only. Implementation of this symmetry is facilitated by that fact that the renormalization of correlation functions in the ϕ–b sector does not involve the ghost renormalization because the ghost propagators are retarded and cannot form closed loops, apart from the tadpole term which is local. (The tadpole term vanishes with dimensional regularization). The t-reversal symmetry leaves no room for a drift force that is not a gradient, so the "gauge function" must remain of the form $b_j + \frac{\partial S}{\partial \phi_j}$, where S is a renormalized action. Finally, BRST symmetry is used to determine the ghost part of the renormalized action from the ϕ–b part. We conclude that the combination of both time-reversal symmetry and BRST invariance implies that the renormalized $(d+1)$-dimensional theory remains equivalent to a local d-dimensional theory on a time-slice.

Note. Because the time-reversal transformation $b(t) \to -b(-t) - \frac{\partial S}{\partial \phi}$, is non linear, one must introduce a source term $\int dt N_i \frac{\partial S}{\partial \phi_i}$ in order to maintain this symmetry in the context of a renormalizable theory. (For BRST invariance one also introduces a source for $s\frac{\partial S}{\partial \phi_i} = \frac{\partial^2 S}{\partial \phi_i \partial \phi_j}\psi_j$.) The reduced partition function $Z_1(j_\phi, j_b, N) = \exp W_1(j_\phi, j_b, N)$, obtained by setting to 0 all sources with non-zero ghost number, may be expressed in terms of the reduced action $I'_{\rm tot}(\phi, b)$,

$$Z_1(J_\phi, J_b, N) = \int d\phi db \ \exp\left[I'_{\rm tot}(\phi, b) + \int dt \left(j_{\phi_i}\phi_i + j_{b_i} b_i + N_i \frac{\partial S}{\partial \phi_i}\right)\right]. \qquad (3.11)$$

It is easy to see that invariance of $I'_{\rm tot}(\phi, b)$ under the non-linear time reversal transformation $b(t) \to -b(-t) - \frac{\partial S}{\partial \phi}(-t)$ [and $\phi(t) \to \phi(-t)$] implies that the generating functional of connected correlation functions satisfies the symmetry condition

$$W_1(j_\phi, j_b, N) = W_1(j^T_\phi, j^T_b, N^T + j^T_b), \qquad (3.12)$$

where $j^T_\phi(t) = j_\phi(-t)$, $j^T_b(t) = -j_b(-t)$, and $N^T(t) = N(-t)$. However it does not appear that this condition is easily expressed in terms of the reduced effective action $\Gamma_1(\phi, b, N)$ obtained from $W_1(j_\phi, j_b, N)$ by Legendre transformation. The solution is to introduce a source $j_{b'_i}$ for the canonically conjugate field $b'_i = b_i + \frac{1}{2}\frac{\partial S}{\partial \phi_i}$, which obeys the elementary transformation law $b'(t) \to -b'(-t)$. The corresponding reduced partition function is defined by

$$\begin{aligned} Z'(j_\phi, j_{b'}, N) &= \exp W'(j_\phi, j_{b'}, N) \\ &\equiv \int d\phi db' \ \exp\left[I'_{\rm tot}(\phi, b') + \int dt \ (j_{\phi_i}\phi_i + j_{b'_i}b'_i) + N_i \frac{\partial S}{\partial \phi_i}\right], \end{aligned} \tag{3.13}$$

where $I'_{\rm tot}(\phi, b')$ is the local action (3.3). The two generating functionals are related by $W'(j_\phi, j_{b'}, N) = W_1(j_\phi, j_{b'}, N + \frac{1}{2}j_{b'})$, so they provide the same information. Invariance of $I'_{\rm tot}(\phi, b')$ under the elementary time-reversal transformations of ϕ and b' implies that $W'(j_\phi, j_{b'}, N)$ satisfies the elementary and standard time-reflection condition $W'(j_\phi, j_{b'}, N) = W'(j_\phi^T, j_{b'}^T, N^T)$, where $j_\phi^T(t) = j_\phi(-t)$, $j_{b'}^T(t) = -j_{b'}(-t)$, and $N^T(t) = N(-t)$. Consequently the reduced effective action $\Gamma'(\phi, b', N)$, obtained by Legendre transformation from $W'(j_\phi, j_{b'}, N)$, also satisfies the elementary and standard time-reflection condition

$$\Gamma'(\phi, b', N) = \Gamma'(\phi^T, b'^T, N^T)\,, \tag{3.14}$$

where $\phi^T(t) = \phi(-t)$, $b'^T(t) = -b'(-t)$ and $N^T(t) = N(-t)$. In each order of perturbation theory, the divergent terms in the ϕ–b sector are local contributions to $\Gamma'(\phi, b', N)$ which must satisfy this standard symmetry condition.

4. S-matrix

We have shown above that the correlation functions of the d-dimensional theory are obtained from the $(d+1)$-dimensional theory by

$$G_d^{(n)}(x_1, \ldots,\ x_n) = G_{d+1}^{(n)}(t_1, x_1, \ldots,\ t_n, x_n)|_{t_1=\cdots,t_n=0}\,, \tag{4.1}$$

where $G_d^{(n)}$ and $G_{d+1}^{(n)}$ are correlators of n ϕ-fields. We write this in 5-dimensional momentum space as

$$G_d^{(n)}(p_1, \ldots, p_n) = (2\pi)^{-n+1} \int \prod_{i=1}^n dE_i\ \delta\left(\sum_{i=1}^n E_i\right) G_{d+1}^{(n)}(E_1, p_1, \ldots, E_n, p_n)\,. \tag{4.2}$$

Here the δ-function $\delta(\sum_i E_i)$ expresses invariance under translation in the (d+1)-th time. We have removed a momentum conserving δ-function $\delta^d(\sum_{i=1}^n p_i)$ from both sides, so the p_i are understood to be constrained by $\sum_{i=1}^n p_i = 0$. The $(d+1)$-dimensional action allows one to derive Feynman rules for the $(d+1)$-dimensional correlation functions in the euclidean region, where the p_μ are real for $\mu = 1, \ldots, d$. In the following we shall deal with the connected components of the correlators and of the S-matrix which alone may be continued analytically. However we shall not trouble to write out the expansion in terms of connected components explicitly.

To obtain the physical S-matrix, the $(d+1)$-dimensional connected correlators $G^{(n)}_{d+1}(E_i, p_i)$ must be continued from the euclidean region in the $p_{i,\mu}$, where $\mu = 1, \ldots, d$, to the on-shell Minkowskian region, $p_i^2 = -m_i^2$, for real positive mass, $m_i \geq 0$, by continuation to imaginary values of the d-th component $p_{i,d}$. We shall see below that the region of analyticity of the $(d+1)$-dimensional correlators in the $p_{i,\mu}$, is at least as great as it is for the d-dimensional correlators, so this continuation is always possible. According to the LSZ method in d dimensions, the S-matrix is obtained from the correlation function in momentum space by amputating each leg namely by multiplying by $X_i \equiv (p_i^2 + m_i^2)$ and going on mass shell. Thus the S-matrix element with n external legs is given by

$$
\begin{aligned}
S^{(n)} &= \lim_{X_i \to 0} \prod_{i=1}^{n} X_i \; G^{(n)}_d(p_i)\,, \\
S^{(n)} &= (2\pi)^{-n+1} \lim_{X_i \to 0} \prod_{i=1}^{n} X_i \int \prod_{i=1}^{n} dE_i \; \delta\left(\sum_{i=1}^{n} E_i\right) G^{(n)}_{d+1}(E_i, p_i)\,.
\end{aligned} \tag{4.3}
$$

We wish to express $S^{(n)}$ directly in terms of the truncated $(d+1)$-dimensional correlation functions $\Gamma^{(n)}_{d+1}$ which are related to the $G^{(n)}_{d+1}$ by

$$
(G^{(n)}_{d+1})_{\phi_1,\ldots,\,\phi_n} = D_{\phi_1\alpha_1} \cdots \; D_{\phi_n\alpha_n} \; (\Gamma^{(n)}_{d+1})_{\alpha_1,\ldots,\alpha_n}\,, \tag{4.4}
$$

where $\alpha_i \equiv (\phi_i, b_i)$ is a 2-valued index. We write this in matrix notation as

$$
G^{(n)}_{d+1} = \prod_i D_i \; \Gamma^{(n)}_{d+1}\,, \tag{4.5}
$$

where it is understood that the row label of D_i is ϕ. Here $D_{\alpha\beta} \equiv D_{\alpha\beta}(E,p)$ is the 2×2 propagator matrix,

$$
\begin{pmatrix} D_{\phi\phi}(E,p) & D_{\phi b}(E,p) \\ D_{b\phi}(E,p) & 0 \end{pmatrix} \tag{4.6}
$$

and $D_{bb}(E,p) = 0$ by s-invariance, as shown in (4.10) below. This gives

$$
S^{(n)} = (2\pi)^{-n+1} \lim_{X_i \to 0} \int \prod_{i=1}^{n} dE_i \; \delta\left(\sum_{i=1}^{n} E_i\right) \prod_{i=1}^{n} (X_i D_i) \; \Gamma^{(n)}_{d+1}(E_i, p_i)\,, \tag{4.7}
$$

Because each leg is multiplied by $X_i \to 0$, only a singular part will survive.

We consider the case of scalar particles with kernel $K = 1$ in which case the free propagators in "energy" and momentum space are given by

$$
\begin{aligned}
(D_0)_{\phi\phi} &= 2[E^2 + (p^2 + m^2)^2]^{-1}\,, \qquad (D_0)_{\phi b} = -(p^2 + m^2 + iE)^{-1}\,, \\
(D_0)_{b\phi} &= -(p^2 + m^2 - iE)^{-1}\,, \qquad (D_0)_{bb} = 0\,.
\end{aligned} \tag{4.8}
$$

The external "energies" are real and non zero, so the domain of analyticity in the external momentum components p_μ is greater than for the corresponding d-dimensional euclidean Feynman graphs, and thus the continuation from the euclidean to the on-shell Minkowskian region is possible. We will also show below that the relevant graphs are analytic in every upper-half E_i-plane, where E_i is the "energy" of the i-th external line.

To proceed further we use simple properties of truncated diagrams which are a direct consequence of the fact that the bulk action is s-exact. From the elementary identities

$$0 = \left\langle s\left(\bar{\psi}(t,x)b(0,0)\right)\right\rangle = \langle b(t,x)b(0,0)\rangle\,,$$
$$0 = \left\langle s\left(\phi(t,x)\bar{\psi}(0,0)\right)\right\rangle = \left\langle \psi(t,x)\bar{\psi}(0,0)\right\rangle + \langle \phi(t,x)b(0,0)\rangle \tag{4.9}$$

we obtain

$$D_{bb}(t,x) = 0\,, \qquad D_{\phi b}(t,x) = -D_{\psi\bar{\psi}}(t,x)\,. \tag{4.10}$$

Moreover the ghost propagator $D_{\psi\bar{\psi}}(t,x)$ is retarded, as shown in appendix A, and thus so is $D_{\phi b}$,

$$D_{\phi b}(t,x) = -D_{\psi\bar{\psi}}(t,x) = 0\,, \qquad \text{for } t<0\,. \tag{4.11}$$

There are no closed ghost loops, apart from a tadpole term, because the ghost propagator is retarded.

We now use these properties to show that every non-zero truncated diagram contains at least one external b-line. Note from the action (2.7) with $K = 1$ that each vertex contains precisely one b-line. Now consider a fixed vertex of the diagram. Call it V_1. Either its b-line is external, and if so the assertion is true, or if not, this b-line connects to another vertex, call it V_2, by a $(D_0)_{\phi b}$ propagator because the $(D_0)_{bb}$ propagator vanishes. We now repeat the argument for V_2. The vertex V_2 contains one b-line which is either external or connected to a third vertex V_3 by a $(D_0)_{\phi b}$ propagator. (See figure 1.) Because the $(D_0)_{\phi b}$ propagators are all retarded, this sequence follows the direction of increasing time, and each vertex in the sequence is different. We conclude that every finite diagram has at least one external b-line, as asserted. One may also prove this algebraically using the Ward identities for s-invariance. They imply that the one-particle irreducible correlation functions have this property, from which it follows that the truncated correlation functions do.

The above argument also shows that every truncated diagram Γ_{d+1} vanishes unless the largest time is associated to an external b-line. In particular, if there is only a single external b-line, then it must be associated to the largest time. In fact we will see shortly that only truncated graphs with a single b-line contribute to the S-matrix. Consider one such graph, and let the ϕ-fields be at $\phi(x_i,t_i)$, for $i = 1,\ldots,n-1$, and let the b-field be taken at the origin $b(0,0)$. This graph vanishes if any one of the t_i is positive. Therefore, in momentum space, the graph in question is analytic in every upper-half E_i-plane.

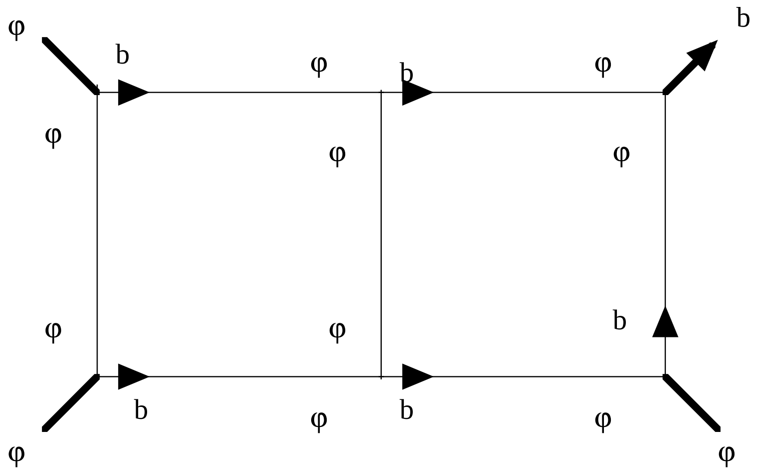

Figure 1: Every vertex has one emerging b-line. One may contunously follow $(D_0)_{\phi b}$ propagators from every vertex of a truncated diagram to an external b-line. The arrows follow the direction of increasing time.

Let $\Gamma_{d+1}^{(n-p,p)}$ be the part of $\Gamma_{d+1}^{(n)}$ with p external b-lines and $(n-p)$ external ϕ-lines, and call $S^{(n-p,p)}$ the corresponding contribution to $S^{(n)}$, according to (4.7). We have just shown that $\Gamma_{d+1}^{(n,0)} = 0$, so $S^{(n)} = \sum_{p=1}^{n} S^{(n-p,p)}$. We will see shortly that $S^{(n-p,p)} = 0$ for $p \geq 2$, so

$$S^{(n)} = S^{(n-1,1)}\,. \tag{4.12}$$

We next evaluate $S^{(n-1,1)}$ which comes from $\Gamma_{d+1}^{(n-1,1)}$. Any single one of the n external lines of $\Gamma_{d+1}^{(n-1,1)}$ may be the b-line, and we have $\Gamma_{d+1}^{(n-1,1)} = \sum_{j=1}^{n} \Gamma_{d+1}^{(n-1,1,j)}$, where $\Gamma_{d+1}^{(n-1,1,j)}$ is the truncated correlation function where the j-th external line is a b-line, and all other lines are ϕ-lines. Correspondingly we have the decomposition

$$S^{(n)} = S^{(n-1,1)} = \sum_{i=1}^{n} S^{(n-1,1,j)}\,. \tag{4.13}$$

We now evaluate

$$\begin{aligned} S^{(n-1,1,j)} = (2\pi)^{-n+1} \int dE_j \lim_{X_j\to 0} X_j D_{\phi b}(E_j, p_j) \times \\ \times \int \prod_{i\neq j} dE_i \lim_{X_i \to 0} X_i\ D_{\phi\phi}(E_i, p_i)\ \delta\left(\sum_{i=1}^{n} E_i\right) \Gamma_{d+1}^{(n-1,1,j)}\,, \end{aligned} \tag{4.14}$$

where $X_i = p_i^2 + m_i^2$. Integration with respect to E_j eliminates the δ-function, $\int dE_j \delta(\sum_{i=1}^{n} E_i) = 1$, and we have the constraint that the "energy" is conserved. The remaining external lines, for $i \neq j$ are all ϕ-lines. We must calculate $\lim_{X_i\to 0} X_i D_{\phi\phi}$ (E_i, X_i), where $X_i = p_i^2 + m^2$. Let us provisionally make the simplifying assumption that $D_{\phi\phi}(E_i, X_i)$ and $D_{\phi b}(E_j, X_j)$ are the free propagators (4.8). In this case we have the simple and useful limit

$$\lim_{X_i \to 0} X_i D_{\phi\phi}(E_i, X_i) = \lim_{X_i \to 0} \frac{2X_i}{E_i^2 + X_i^2} = 2\pi\delta(E_i) \,. \tag{4.15}$$

We use this identity for all $i \neq j$, so when we integrate on E_i we have $\int dE_i \delta(E_i) = 1$, and everywhere else we set $E_i = 0$ and $X_i = 0$ for $i \neq j$. By conservation of "energy" we also have $E_j = 0$. We have provisionally assumed that $D_{\phi b}(E_j, X_j)$ is the free propagator (4.8), so

$$\lim_{X_j \to 0} X_j D_{\phi b}(E_j, p_j)|_{E_j=0} = \lim_{X_j \to 0} \left(\frac{X_j}{X_j + iE_j}|_{E_j=0} \right) = 1 \,, \tag{4.16}$$

and everywhere else we set $X_j = 0$. This gives

$$S^{(n-1,1,j)} = \Gamma_{d+1}^{(n-1,1,j)}|_{E_i=0, X_i=0} \tag{4.17}$$

for all $i = 1, \ldots, n$. Since the external b-leg may be any one of the legs we obtain the remarkably simple formula

$$S^{(n)} = S^{(n-1,1)} = \sum_{j=1}^{n} \Gamma_{d+1}^{(n-1,1,j)}|_{E_i=0, X_i=0} \,. \tag{4.18}$$

We must correct the above argument by use of the exact $D_{\phi\phi}(E, p)$ and $D_{\phi b}(E, p)$ propagators instead of the free ones. The exact propagator is a 2×2 matrix which is the inverse of the matrix

$$\begin{pmatrix} 0 & \Gamma_{\phi b}(E, p) \\ \Gamma_{b\phi}(E, p) & \Gamma_{bb}(E, p) \end{pmatrix} \tag{4.19}$$

where $\Gamma_{\phi b}(E, p) = \Gamma_{b\phi}(-E, -p) = \Gamma_{b\phi}(-E, p), = \Gamma_{b\phi}^*(E, p)$ namely

$$\begin{aligned} D_{\phi b}(E, p) &= \frac{1}{\Gamma_{b\phi}(E, p)} \,, \\ D_{\phi\phi}(E, p) &= \frac{-\Gamma_{bb}(E, p)}{|\Gamma_{b\phi}(E, p)|^2} \,. \end{aligned} \tag{4.20}$$

These quantities have the perturbative expansions

$$\begin{aligned} -\Gamma_{bb} &= 1 + O(g^2) \,, \\ -\Gamma_{b\phi} &= iE + p^2 + m_0^2 + O(g^2) \,, \end{aligned} \tag{4.21}$$

where m_0 is the bare mass, as one reads off from $\Gamma_0 = -I_0$, where I_0 is the zeroth order part of the action I. To zeroth order $\Gamma_{b\phi}(E, p)$ vanishes at $iE + p^2 + m_0^2 = 0$. We suppose that the exact quantity $\Gamma_{b\phi}(E, p)$ vanishes at

$$iE + f(p^2) = 0 \,, \tag{4.22}$$

where $f(p^2)$ is an arbitrary real function. Then we have

$$-\Gamma_{b\phi}(E,p) = [iE + f(p^2)]R(E,p^2)\,, \tag{4.23}$$

where $R(E,p^2)$ is regular at $iE + f(p^2) = 0$. We suppose also that $f(p^2)$ vanishes at $p^2 = -m^2$, where, naturally, m is the renormalized mass. We define $X \equiv p^2 + m^2$, and we have

$$f(p^2) = X r(X)\,, \tag{4.24}$$

where $r(X)$ is a function that is regular at $X = 0$. This gives

$$\begin{aligned}
\Gamma_{b\phi}(E,p) &= [iE + Xr(X)]R(E,X)\,,\\
D_{\phi b}(E,p) &= \frac{1}{[iE + Xr(X)]R(E,X)}\,,\\
D_{\phi\phi}(E,p) &= \frac{-\Gamma_{bb}(E,X)}{|R(E,X)|^2[E^2 + X^2\, r^2(X)]}\,.
\end{aligned} \tag{4.25}$$

Instead of (4.15) we now obtain

$$\lim_{X_i\to 0} X_i D_{\phi\phi}(E_i, X_i) = 2\pi Z_{\phi\phi}\delta(E_i)\,. \tag{4.26}$$

where $Z_{\phi\phi}$ is the renormalization constant

$$Z_{\phi\phi} = \frac{-\Gamma_{bb}(0,0)}{|R(0,0)|^2 r(0)}\,, \tag{4.27}$$

and instead of (4.16) we obtain

$$\lim_{X_j\to 0} X_j D_{\phi b}(E_j, p_j)|_{E_j=0} = Z_{\phi b}\,, \tag{4.28}$$

where

$$Z_{\phi b} = \frac{1}{r(0)R(0,0)}\,. \tag{4.29}$$

Thus, apart from renormalization constants, we obtain (4.18).

There remains only to show that the contribution to the S-matrix from $\Gamma_{d+1}^{(n-p,p)}$ vanishes for $p \geq 2$, namely if there is more than one external b-line on $\Gamma_{d+1}^{(n)}$. But this is immediate because we have

$$\lim_{X\to 0} X D_{\phi b}(E,p) = \lim_{X\to 0} \frac{X}{[iE + Xr(X)]R(E,X)} = 0\,, \tag{4.30}$$

since, in contradistinction to (4.16), E is not contrained to be 0.

We conclude that the S-matrix is given by (4.18) corrected by the renormalization constants, namely

$$S^{(n)}(p_1,\ldots,p_n) = Z_{\phi b}\; Z_{\phi\phi}^{n-1} \sum_{j=1}^{n} \Gamma_{d+1}^{(n-1,1,j)}(E_1,p_1,\ldots,E_n,p_n)|_{E_i=0,X_i=0}\,. \tag{4.31}$$

Thus in the bulk formalism, the S-matrix is obtained from $\Gamma_{d+1}^{(n-1,1,j)}$, the truncated correlation function with exactly one external b-line (in all possible positions j) by going "on-shell", namely by setting $E_i = 0$ and $X_i = 0$ for all i.

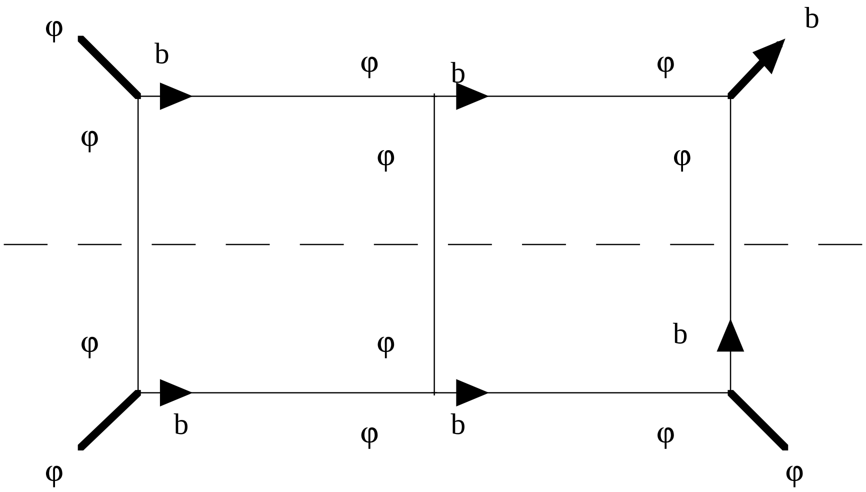

Figure 2: A cut diagram, with one $(D_0)_{\phi b}$ intermediate line.

5. Landau-Cutkosky rules and perturbative unitarity

The Landau-Cutkosky rules follow from properties of integrals of the form $\int dx f(x,z)$, where x and z are sets of complex variables, corresponding respectively to internal and external (d+1)-momenta. These rules hold wherever these variables may be continued. Thus to derive the Landau-Cutkosky rules, we may use Feynman rules for the euclidean correlators, with the understanding that a continuation in external momenta is made to the Minkowski region, as discussed in the last section.

Consider an on-shell connected and truncated graph which contributes to the S-matrix. As we have just shown, it has an arbitrary number of external ϕ-lines, and precisely one external b-line. We cut it into two subgraphs across n intermediate lines, with incoming and outgoing particles on opposite sides of the cut. Let p_i and E_i, for $i = 1, \dots, n$, be the "momentum" and "energy" of the i-th intermediate line. The total d-momentum $P = \sum_{i=1}^{n} p_i$ entering through the incoming lines is fixed, as is the total "energy" which vanishes, since it is on-shell, $0 = \sum_{i=1}^{n} E_i$. Thus the integration associated with the cut lines is

$$\prod_{i=1}^{n} d^d p_i dE_i \; \delta^d\left(\sum_{i=1}^{n} p_i - P\right) \delta\left(\sum_{i=1}^{n} E_i\right) = \prod_{i=1}^{n-1} d^d p_i dE_i \,. \tag{5.1}$$

The one external b-line of the S-matrix falls on one side of the cut, and we consider a particular vertex on the other side of the cut. As was noted above, we may continuously follow $(D_0)_{\phi,b}$ propagators from any vertex of the graph to the one external b-line. See figure 2. Therefore at least one of the intermediate cut lines is a $(D_0)_{\phi,b}$ propagator. Consider first the case when $n-1$ intermediate cut lines are $(D_0)_{\phi,\phi}$ propagators

$$2[E_i^2 + (p_i^2 + m_i^2)^2]^{-1} = 2(p_i^2 + m_i^2 + iE_i)^{-1}(p_i^2 + m_i^2 - iE_i)^{-1} \tag{5.2}$$

for $i = 1, \ldots, (n-1)$, while the n-th line is a $(D_0)_{\phi,b}$ propagator

$$-(p_n^2 + m_n^2 + iE_n)^{-1} = -\left[\left(P - \sum_{i=1}^{n-1} p_i\right)^2 + m_n^2 + i\left(-\sum_{i=1}^{n-1} E_i\right)\right]^{-1} . \tag{5.3}$$

Each subgraph has exactly one external b-line, so each contributes to an S-matrix element by the reduction formula of the preceding section. We wish to find the location of the leading singularity associated with a pinching of the integration (5.1) aassociated with these lines, and to calculate the discontinuity across it.

For this purpose we combine all the denominators that correspond to cut lines by means of Feynman's auxiliary parameters, using the factorized form (5.2) for the $(D_0)_{\phi,\phi}$ propagators. This gives the overall denominator D^n, where

$$\begin{aligned} D(p_i, E_i, \alpha_i, \beta_i, \alpha_n) \equiv & \sum_{i=1}^{n-1}[\alpha_i(p_i^2 + m_i^2 + iE_i) + \beta_i(p_i^2 + m_i^2 - iE_i)] + \\ & + \gamma\left[\left(P - \sum_{i=1}^{n-1} p_i\right)^2 + m_n^2 - \sum_{i=1}^{n-1} iE_i\right] . \end{aligned} \tag{5.4}$$

According to the Landau rules, a pinching singularity occurs where D and all its first derivatives vanish,

$$\frac{\partial D}{\partial \alpha_i} = p_i^2 + m_i^2 + iE_i = 0\,, \tag{5.5}$$

$$\frac{\partial D}{\partial \beta_i} = p_i^2 + m_i^2 - iE_i = 0\,, \tag{5.6}$$

$$\frac{\partial D}{\partial \gamma} = \left(P - \sum_{i=1}^{n-1} p_i\right)^2 + m_n^2 - \sum_{i=1}^{n-1} iE_i = 0\,, \tag{5.7}$$

$$\frac{\partial D}{\partial p_i} = 2(\alpha_i + \beta_i)p_i + 2\gamma\left(\sum_{i=1}^{n-1} p_i - P\right) = 0\,, \tag{5.8}$$

$$\frac{\partial D}{\partial E_i} = i(\alpha_i - \beta_i - \gamma) = 0\,, \tag{5.9}$$

for $i = 1, \ldots, (n-1)$. From (5.5) and (5.6) we obtain $E_i = 0$, and $p_i^2 + m_i^2 = 0$ for $i = 1, \ldots, (n-1)$. Together with (5.7), this gives $(P - \sum_{i=1}^{n-1} p_i)^2 + m_n^2 = 0$. So all n lines are on the mass-shell, and all n energies vanish. This implies that $D = 0$. From (5.9) we obtain $\beta_i = \alpha_i - \gamma$. We substitute this into (5.8) and obtain $(2\alpha_i - \gamma)p_i + \gamma(\sum_{i=1}^{n-1} p_i - P) = 0$. We change variable from α_i to $\alpha_i' \equiv 2\alpha_i - \gamma$, and we write $\alpha_n' \equiv \gamma$, so the last equation reads $\alpha_i' p_i + \alpha_n'(\sum_{i=1}^{n-1} p_i - P) = 0$.

We conclude that the conditions for a singularity are the conditions $E_i = 0$, plus the standard Landau equations for the momenta p_i and p_n, which imply that all momenta are on the mass shell. This is the on-shell condition of the $(d+1)$-dimensional theory which we found for the S-matrix. The singularities occur at Minkowksian values of the p_i, and the singularity in P occurs at the physical threshold $P^2 = (\sum_{i=1}^{n} m_i)^2$.

To evaluate the discontinuity across the cut that begins at $P^2 = (\sum_{i=1}^{n} m_i)^2$, we integrate over the E_i, for $i = 1, \ldots, (n-1)$, by closing the contour in the upper half-plane. In this way we pick up the 2π times the residue of the first pole in (5.2), at $E_i = i(p_i^2 + m_i^2)$. As a result, each propagator $(D_0)_{\phi\phi}(E_i, p_i)$ is replaced by $(p_i^2 + m_i^2)^{-1}$ for $i = 1, \ldots, (n-1)$, and the propagator $(D_0)_{\phi b}(E_n, p_n)$ is replaced by $[(P - \sum_{i=1}^{n-1} p_i)^2 + m_n^2 + \sum_{i=1}^{n-1}(p_i^2 + m_i^2)]^{-1}$. We have found from our Landau rules that the leading singularity occurs when all these propagators vanish, namely at $p_i^2 + m_i^2 = 0$ for $i = 1, \ldots, n$. The problem of calculating the discontinuity has now been reduced to the familiar d-dimensional problem, and we know from the d-dimensional Cutkosky rules that the leading discontinuity is obtained by replacing each propagator $(p_i^2 + m_i^2)^{-1}$ by $2\pi\delta(p_i^2 + m_i^2)$. The net result is that the leading discontinuity is obtained by replacing each propagator $(D_0)_{\phi\phi}(E_i, p_i)$ by $d\pi^2\delta(E_i)\delta(p_i^2 + m_i^2)$ for $i = 1, \ldots, (n-1)$, and the propagator $(D_0)_{\phi b}(E_n, p_n)$ by $2\pi\delta(p_n^2 + m_n^2)$. This gives the discontinuity required for the S-matrix to be unitarity.

Suppose now that more than one intermediate cut line is a $(D_0)_{\phi b}$ propagator. Then the Landau rules that correspond to (5.5) through (5.9) are such that not all intermediate momenta are on mass shell. The corresponding pinching singularities cannot contribute to the unitarity equation.

6. Ward Identities for a global symmetry

We have shown that the topological $(d+1)$-dimensional theory reproduces the standard d-dimensional theory in a time slice. It follows that if the d-dimension action S has a global invariance, the Ward identities of the d-dimensional theory hold on a time-slice of the $(d+1)$-dimensional theory. However it is possible to prove them directly in $d+1$ dimensions. In the derivation which we present here, the Noether current in $d+1$ dimensions does not appear at all. However in appendix B we give an alternative derivation of the Ward identities that relies on the $(d+1)$-dimensional Ward identity, with an integration in t on a very thin slice. As in the derivation of the SD equations, BRST-invariance plays an essential role. At the end of this section we explain how the anomalies of the d-dimensional theory arise in the bulk formulation.

Suppose that the infinitesimal transformation $\delta\phi = \epsilon\lambda\phi$ is a symmetry of the d-dimensional euclidean action S. Here ϵ is an infinitesimal constant, and λ is a numerical matrix that satisfies $\mathrm{Tr}\,\lambda = 0$. In this case Noether's theorem holds, namely $\frac{\delta S}{\delta\phi(x)}\lambda\phi(x) = -\partial_\mu j_\mu(x)$, where j_μ, for $\mu = 1, \ldots, d$, is the Noether current. This property of the action leads to the Ward identity that holds in euclidean quantum field theory

$$0 = \left\langle \left(\frac{\delta \mathcal{O}}{\delta \phi(x)} \lambda \phi(x) + \mathcal{O} \partial_\mu j_\mu(x) \right) \right\rangle_d . \tag{6.1}$$

To establish this Ward identity in the bulk quantization, we start from the identity

$$0 = \int \mathcal{D}\phi \mathcal{D}b \mathcal{D}\psi \mathcal{D}\bar{\psi} \ \mathcal{O}(\phi(0)) \ (\lambda\phi(0))_k K^{-1}_{ki}(\phi(0)) \ \frac{\delta \exp I_{\rm tot}}{\delta b_i(0)} , \tag{6.2}$$

where again the discrete index i represents x_μ and all internal indices. We repeat the steps of the derivation of the SD equations and obtain

$$0 = \left\langle \ \mathcal{O} \frac{\partial S}{\partial \phi_k} (\lambda\phi)_k - \frac{\partial \mathcal{O}}{\partial \phi_k} (\lambda\phi)_k \right\rangle_{d+1} , \tag{6.3}$$

where all fields are evaluated at $t = 0$. In continuum notation this reads

$$0 = \left\langle \left(\mathcal{O} \frac{\delta S}{\delta \phi(x)} \lambda \phi(x) - \frac{\delta \mathcal{O}}{\delta \phi(x)} \lambda \phi(x) \right) |_{\phi(x) = \phi(x,0)} \right\rangle_{d+1} . \tag{6.4}$$

On using Noether's theorem $\frac{\delta S}{\delta \phi(x)} \lambda \phi(x) = -\partial_\mu j_\mu(x)$ for the first term we recover the desired Ward identity.

It may happen that the correlator $\langle \partial_\mu j_\mu(x,t) \ \mathcal{O}[\phi(x,0)] \rangle_{d+1}$ is discontinuous at $t = 0$, so that its value at $t = 0$ is ambiguous. In fact, as was shown in [9], such a discontinuity does occur for $\mathcal{O}$ of the form $\mathcal{O} = j_a(y,0) j_b(z,0)$ when the 3 currents correspond to the familiar triangle graph that exhibits the chiral anomaly. In this case the above Ward identity involving the d-dimensional Noether current j_μ is anomalous. This happens even though the Ward identity for $\mathcal{O} = \mathcal{O}[\phi]$

$$0 = \left\langle \frac{\delta^{d+1} \mathcal{O}}{\delta \phi(x,t)} \lambda \phi(x,t) - \mathcal{O} \partial_M K_M \right\rangle_{d+1} , \tag{6.5}$$

is not anomalous, where K_M for $M = 1, \ldots, (d{+}1)$, is the $(d{+}1)$-dimensional Noether current of the bulk action $I_{\rm tot}$. Here it is assumed that $I_{\rm tot}$ is invariant under the same symmetry as the euclidean action S provided that ϕ, ψ, $\bar{\psi}$, and b are transformed appropriately. The currents j_μ and K_M are different because S and $I_{\rm tot}$ are different. The origin of the anomaly of j_μ in bulk quantization comes from a discontinuity of the correlators at equal time. It is perfectly admissible that the $(d+1)$-dimensional Noether current K_M is conserved, while the d-dimensional Noether current j_μ is not.

7. Conclusion

We have shown the equivalence of the topological $(d+1)$-dimensional formulation of quantum field theory to the standard d-dimensional one for a theory of non-gauge type. Our method does not rely on the relaxation of a stochastic process and applies equally well to bose and to fermi-dirac fields fields. It thus provides a more general framework than stochastic quantization. It is also a more powerful one, for we are able to show the stability of the topological $(d+1)$-dimensional action that is equivalent to a local d-dimensional theory. We also expressed the physical S-matrix directly in terms of the truncated correlation functions of ϕ and its conjugate field b of the $(d+1)$-dimensional theory. Since this approach goes beyond the conceptual framework of stochastic quantization and does not rely on the relaxation hypothesis, we call it "bulk" quantization. In a forthcoming publication we shall show that this approach allows one to overcome the Gribov problem because, strictly speaking, one does not really fix a gauge at all. Instead the infinities associated with the gauge modes are eliminated by making a gauge transformation in the functional integral.

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A. Ghost propagator

By (2.7), the dependence of the action I on the ghosts $\bar{\psi}$ and ψ is of the bilinear form

$$I_{\text{gh}} = -\int dt \; \bar{\psi}_i \left[\delta_{ij}\frac{\partial}{\partial t} + L_{ij}(\phi, b)\right]\psi_j \,, \tag{A.1}$$

where L is given in (2.8). To calculate the ghost propagator with this action we use the identity

$$\begin{aligned} 0 &= \int d\psi d\bar{\psi} \; \frac{\delta}{\delta\bar{\psi}_i(t)}[\bar{\psi}_k(u)\exp I_{\text{gh}}] \\ &= \int d\psi d\bar{\psi} \left[\delta_{ik}\delta(t-u) + \bar{\psi}_k(u)\left(\frac{\delta_{ij}\partial}{\partial t} + L_{ij}\right)\psi_j(t)\right]\exp I_{\text{gh}} \,. \end{aligned} \tag{A.2}$$

Thus, with due attention paid to the anti-commutation of the ghost fields, the ghost propagator in fixed ϕ and b fields

$$G(t,u;\phi,b)_{jk} \equiv N\int d\psi d\bar{\psi} \; \psi_j(t)\bar{\psi}_k(u)\exp I_{\text{gh}} \tag{A.3}$$

satisfies

$$\left[\frac{\delta_{ij}\partial}{\partial t} + L_{ij}(\phi(t), b(t))\right]G_{jk}(t,u;\phi,b) = \delta_{ik}\delta(t-u) \,. \tag{A.4}$$

For the cases of interest $\partial/\partial t + L$ is a parabolic operator. Indeed L is of the form $L(\phi, b) = L_0 + L_{\text{int}}(\phi, b)$, where L_0 is independent of the fields ϕ and b, and is a positive second order differential operator in ∂_μ, for $\mu = 1, \ldots, d$. For example,

$L_0 = -\partial^2 + m^2$, for the model (2.2) with $K_{ij} = \delta_{ij}$, where $\partial^2 = \sum_{\mu=1}^d \partial_\mu^2$. (Recall that the discrete index i or j stands for the continuum variables x_μ.) We assume that $L_{\rm int}(\phi, b)$ vanishes with ϕ and b and contains no derivative ∂_μ higher than the first. Because the ghost propagator G is the Green function of a parabolic operator, it is retarded,

$$G(t, u; \phi, b)_{jl} = 0\,, \qquad \text{for } t < u\,, \tag{A.5}$$

(as follows from the integral equation below.) This is a fundamental property of the topological theory for it has as a consequence that all closed ghost loops vanish (except for the tadpole term which is given in (3.7)).

The free Green function $G_{0,ij}(t-u)$ is defined by

$$\left[\frac{\delta_{ij}\partial}{\partial t} + L_{0,ij}\right] G_{0,jk}(t-u) = \delta_{ik}\delta(t-u)\,, \tag{A.6}$$

and is expressed in matrix or operator form by

$$G_{0,ij}(t-u) = \theta(t-u)\exp[-L_0(t-u)]\,. \tag{A.7}$$

It is related to the exact Green function (A.4) by

$$G_{il}(s, u; \phi, b) = G_{0,il}(s-u) - \int_u^s dt\; G_{0,ij}(s-t)L_{{\rm int},jk}(t)G_{kl}(t, u, \phi, b)\,. \tag{A.8}$$

These expressions vanish unless $s \geq t \geq u$. The second term vanishes for $s \to u$. It follows from the last two equations that for $t \approx u$

$$G(t, u; \phi, b)_{jl} = \delta_{jl}\theta(t-u) + o(t-u)\,, \tag{A.9}$$

where $\theta(t-u)$ is the step function, and $o(t-u)$ vanishes with $t-u$. Moreover a consistent determination of $\theta(0)$ is $\theta(0) = 1/2$.

We conclude that inside the expectation value we may make the replacements

$$\begin{aligned} \psi_j(t)\bar\psi_l(u) &\longrightarrow G(t, u; \phi, b)_{jl}\,, \\ \psi_j(0)\bar\psi_l(0) &\longrightarrow \left(\frac{1}{2}\right)\delta_{jl}\,. \end{aligned} \tag{A.10}$$

B. Alternative proof of Ward identities and conserved currents

We write the action (2.7) in the form $I = \int dt d^dx \mathcal{L}_{d+1}$. We suppose that the lagrangian density $\mathcal{L}_{d+1}$ is invariant under the infinitesimal transformation $\delta\phi = \epsilon\lambda\phi$, $\delta\psi = \epsilon\lambda\psi$, $\delta\bar\psi = -\epsilon\bar\psi\lambda$, and $\delta b = -\epsilon b\lambda$, where ϵ is an infinitesimal constant and λ is a numerical matrix with ${\rm Tr}\,\lambda = 0$. If instead $\epsilon = \epsilon(x, t)$ is an infinitesimal function

JHEP08(2001)016

in $d+1$ dimensions, then according to Noether's theorem, the lagrangian density changes according to

$$\delta\mathcal{L}_{d+1} = \frac{\partial \mathcal{L}_{d+1}}{\partial\, \partial_M A_i}(\lambda A)_i \partial_M \epsilon \equiv K_M \partial_M \epsilon\,, \tag{B.1}$$

under this transformation, where $A_i = (\phi, \psi, \bar{\psi}, b)$ represents the set of all fields, and $\delta A_i = (\lambda A)_i \epsilon$ their transform. Here K_M, for $M = 1, \ldots, (d+1)$, is the Noether current of the $(d+1)$-dimensional topological action (2.7). For this action, the time-component of the Noether current has the particularly simple expression

$$K_{d+1} = s(\bar{\psi}\lambda\phi) = b\lambda\phi - \bar{\psi}\lambda\psi\,. \tag{B.2}$$

Let $\mathcal{O}[\phi]$ be an observable. Upon making the preceding infinitesimal change of variables for arbitrary $\epsilon(x,t)$ in the functional integral, $\langle\mathcal{O}\rangle|_{d+1} = \int \mathcal{D}A\ \mathcal{O} \exp I$ we obtain the $(d+1)$-dimensional Ward identity corresponding to the above symmetry,

$$\begin{aligned} 0 &= \left\langle \left[\frac{\delta^{d+1}\mathcal{O}}{\delta\phi(x,t)} \lambda\phi(x,t) - \mathcal{O}\partial_M K_M \right] \right\rangle_{d+1} \\ &= \int \mathcal{D}A_i(x,t)\ \left[\frac{\delta^{d+1}\mathcal{O}}{\delta\phi(x,t)} \lambda\phi(x,t) - \mathcal{O}\partial_M K_M \right] \exp I\,, \end{aligned} \tag{B.3}$$

where $\mathcal{D}A_i(x,t) \equiv \mathcal{D}\phi(x,t)\mathcal{D}\Psi(x,t)\mathcal{D}\bar{\Psi}(x,t)\mathcal{D}b(x,t)$, and $\delta^{d+1}\mathcal{O}/\delta\phi(x,t)$ designates the functional derivative in $(d+1)$ dimensions.

We specialize to the case where $\mathcal{O}$ is a physical observable, namely it is a functional only of $\phi(x,t)$ on the time slice $t=0$, so $\mathcal{O} = \mathcal{O}[\phi(x,0)] = \mathcal{O}[\phi(x)]|_{\phi(x)=\phi(x,0)}$. For such functionals one easily verifies that the functional derivative may be written

$$\frac{\delta^{d+1}\mathcal{O}}{\delta\phi(x,t)} = \frac{\delta^{d}\mathcal{O}}{\delta\phi(x)}|_{\phi(x)=\phi(x,0)}\ \delta(t)\,. \tag{B.4}$$

Integrate over $\int_{-\epsilon}^{\epsilon} dt$, and let $\epsilon \to 0$ to obtain

$$\begin{aligned} 0 = \int \mathcal{D}A_i(x,t) \left[\left(\frac{\delta^{d}\mathcal{O}}{\delta\phi(x)} \lambda\phi(x) \right) |_{\phi(x)=\phi(x,0)} - \mathcal{O}[K_{d+1}(x,\epsilon) - K_{d+1}(x,-\epsilon)] \right] \times \\ \times\ \exp I\,. \end{aligned} \tag{B.5}$$

The first term is the desired variation of the observable in d dimensions.

We next manipulate $K_{d+1}(x,t)|_{-\epsilon}^{+\epsilon}$ to obtain the Noether current in d dimensions. The second term in (B.2) contributes $(\bar{\psi}\lambda\psi)(x,\epsilon) - (\bar{\psi}\lambda\psi)(x,-\epsilon)$. As shown in appendix A, we may make the replacement $\psi_i(t)\bar{\psi}_j(t) \to \frac{1}{2}\delta_{ij}$ inside the expectation value, so the two ghost terms cancel. The first term of (B.2) contributes, $(b\lambda\phi)(x,\epsilon) - (b\lambda\phi)(x,-\epsilon)$. We apply the time-reversal transformation of section 3 to $(b\lambda\phi)(x,-\epsilon)$, under which $\phi(x,-\epsilon) \to \phi(x,\epsilon)$ and $b(x,-\epsilon) \to -b(x,\epsilon) - \frac{\delta^d S}{\delta\phi(x)}|_{\phi(x)=\phi(x,\epsilon)}$, and which leaves $\mathcal{O}[\phi(x,0)]$ invariant. This gives

$$0 = \int \mathcal{D}A_i(x,t) \left[\left(\frac{\delta^d \mathcal{O}}{\delta\phi(x)} \lambda\phi(x) \right) \Big|_{\phi(x)=\phi(x,0)} - \mathcal{O}\; 2(b\lambda\phi)(x,\epsilon) \right.$$
$$\left. - \mathcal{O} \left(\frac{\delta^d S}{\delta\phi(x)} \lambda\phi(x) \right) \Bigg|_{\phi(x)=\phi(x,\epsilon)} \right] \exp I\,. \tag{B.6}$$

The term involving $b = s\bar{\psi}$ gives a vanishing contribution. For by s-invariance of the action, this term may be replaced by

$$2\bar{\psi}(x,\epsilon) s[\lambda\phi(x,\epsilon)\mathcal{O}] = 2(\bar{\psi}\lambda\psi)(x,\epsilon)\mathcal{O} +$$
$$+ 2(\bar{\psi}\lambda\phi)(x,\epsilon) \int d^d y \frac{\delta^d \mathcal{O}}{\delta\phi(y)}\Bigg|_{\phi(y)=\phi(y,0)} \psi(y,0)\,. \tag{B.7}$$

As shown in appendix A, we may make the replacement $\psi(y,0)\bar{\psi}(x,\epsilon) \to G(y,0;x,\epsilon;\phi,b)$ inside the expectation-value, where $G(y,0;x,\epsilon;\phi,b)$ is the ghost propagator in fixed external ϕ and b fields. However, as is also shown in appendix A, this propagator is retarded, and therefore vanishes for positive ϵ. As noted there, we may replace $(\bar{\psi}\lambda\psi)(x,\epsilon)$ by const $\mathrm{Tr}\,\lambda = 0$.

Finally, we suppose that the transformation $\delta\phi = \epsilon\lambda\phi$ for constant ϵ is a symmetry transformation of the d-dimensional action S. Then by the d-dimensional version of (B.1), we have $[\frac{\delta^d S}{\delta\phi(x)}\lambda\phi(x)]|_{\phi(x)=\phi(x,\epsilon)} = -\partial_\mu j_\mu(x,\epsilon)$, for $\mu = 1,\ldots,d$, where j_μ is the d-dimensional Noether current of the euclidean action S. Thus the Ward identity reads

$$0 = \int \mathcal{D}A_i(x,t) \left[\left(\frac{\delta^d \mathcal{O}}{\delta\phi(x)} \lambda\phi(x) \right) |_{\phi(x)=\phi(x,0)} + \partial_\mu j_\mu(x,\epsilon)\; \mathcal{O}[\phi(x)]|_{\phi(x)=\phi(x,0)} \right] \times$$
$$\times \exp I\,. \tag{B.8}$$

On taking the limit $\epsilon \to 0$, one recovers the d-dimensional Ward identity,

$$0 = \left\langle \left(\frac{\delta^d \mathcal{O}}{\delta\phi(x)} \lambda\phi(x) + \mathcal{O}[\phi(x)]\partial_\mu j_\mu(x) \right) \Bigg|_{\phi(x)=\phi(x,0)} \right\rangle_{d+1}, \tag{B.9}$$

in the $(d+1)$-dimensional topological theory. [The signs of (B.3) and (B.9) are consistent because the weights are $\exp(+I)$ and $\exp(-S)$.]

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